Volterra type and weighted composition operators on weighted Fock spaces

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Abstract. Bounded and compact product of Volterra type integral and composition operators acting between weighted Fock spaces are described. We also estimate the norms of these operators in terms of Berezin type integral transforms on the complex plan $\mathbb{C}$. All our results are valid for weighted composition operators acting on the class of Fock spaces considered under appropriate interpretation of the weights.

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1. Introduction

Given a holomorphic function g on the complex plane $\mathbb{C}$, we define the induced Volterra type integral operator by

$$V_g f(z) = \int_0^z f(w)g'(w)dw.$$

Questions about boundedness, compactness, and other operator theoretic properties of V_g expressed in terms of function theoretic conditions on the symbol g have been a subject of high interest since introduced by Pommerenke [16] in 1997. The operator has especially received considerable attentions following the works of Aleman and Siskakis [2, 3] on Hardy and Bergmann spaces. For more information on the subject, we refer to the surveys in [1, 18] and the related references therein.

Each entire function ψ induces a composition operator defined by $C_\psi f = f(\psi)$ on the space of holomorphic functions on $\mathbb{C}$. Using V_g and C_ψ, we define the product of Volterra type integral and composition operators induced by the pair of symbols (g, ψ) as

$$V_{(g,\psi)} f(z) = (V_g \circ C_\psi) f(z) = \int_0^z f(\psi(w))g'(w)dw$$

and

$$C_{(\psi,g)}f(z) = (C_\psi \circ V_g)f(z) = \int_0^{\psi(z)} f(w))g'(w)dw.$$

If $\psi(z) = z$, then these operators are just the usual Volterra type integral operators V_g. As will be indicated latter, studying the operators $V_{(g,\psi)}$ reduces to studying the composition operator C_ψ between Fock spaces when $|g'(z)|/(1+|z|)$ behaves like a constant function. Recent years have seen a lot of work on operators of these kinds acting on different space of holomorphic functions; for example in [11, 13, 15, 17, 23, 22]. Inspired by the works in [7, 8], S. Ueki [20] characterized the bounded weighted composition operators on the classical Fock space $\mathcal{F}_1^2$ in terms of Berezin type integral transforms. Recently, the author obtained similar results for the operator $V_{(g,\psi)}$ when it acts between the Fock spaces $\mathcal{F}_\alpha^p$ and $\mathcal{F}_\alpha^q$ whenever both exponents p and q are finite [15]. The purpose of the present work is partly to continue that line of research and fill the gap when at least one of the Fock spaces in question is the growth space, i.e, the case when one of the exponents is at infinity. We, in addition, establish some interesting results on the mapping properties of $C_{(\psi,g)}$ and weighted composition operators on Fock spaces. Our work unifies and extends a number of results on Volterra type integral and weighted composition operators; for example in [4, 6, 15, 19, 20, 21]. As a main tool in proving some of our results, we will also characterize the (∞, p) Fock–Carleson measures which is an interesting result in its own right.

For $\alpha > 0$, the classical weighted Fock space $\mathcal{F}_\alpha^p$ consists of entire functions f for which

$$\|f\|_{(p,\alpha)}^p = \frac{\alpha p}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{\alpha p}{2}|z|^2} dm(z) < \infty$$

for $0 < p < \infty$ where dm is the usual Lebesgue measure on $\mathbb{C}$, and for $p = \infty$,

$$\|f\|_{(\infty,\alpha)} = \sup_{z\in\mathbb{C}} |f(z)| e^{-\frac{\alpha}{2}|z|^2} < \infty.$$

The space $\mathcal{F}_\alpha^2$ is, in particular, a reproducing kernel Hilbert space with kernel and normalized kernel functions respectively $K_{(w,\alpha)}(z) = e^{\alpha\langle z,w\rangle}$ and $k_{(w,\alpha)}(z) = e^{\alpha\langle z,w\rangle} - \alpha|w|^2/2$.

Our results are expressed in terms of the functions

$$B_{(\psi,\alpha)}^\infty(|g|)(z) = \frac{|g'(z)|}{1+|z|} e^{\frac{\alpha}{2}\left(|\psi(z)|^2 - |z|^2\right)}, \quad M_{(\psi,\alpha)}^\infty(|g|)(z) = \frac{|g'(\psi(z))| e^{\frac{\alpha}{2}\left(|\psi(z)|^2 - |z|^2\right)}}{(1+|z|)(|\psi'(z)|^{-1})}$$

and Berezin type integral transforms

$$B_{(\psi,\alpha)}(|g|^p)(w) = \int_{\mathbb{C}} \frac{|k_{(w,\alpha)}(\psi(z))g'(z)|^p}{\left((1+|z|)e^{\frac{\alpha}{2}|z|^2}\right)^p} dm(z)$$

and

$$M_{(\psi,\alpha)}(|g|^p)(w) = \int_{\mathbb{C}} \frac{|k_{(w,\alpha)}(\psi(z))g'(\psi(z))\psi'(z)|^p}{\left((1+|z|)e^{\frac{\alpha}{2}|z|^2}\right)^p} dm(z).$$

A word on notation: The notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \leq CV(z)$ holds for all z in the set in question, which may be a Hilbert space or a set of complex numbers. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

2. The main results

2.1. Bounded and compact $V_{(g,\psi)}$ and $C_{(\psi,g)}$.

In this subsection, we formulate the main results on the bounded and compact mapping properties of the operators $V_{(g,\psi)}$ and $C_{(\psi,g)}$.

Theorem 2.1. *Let $0 < p \leq \infty$ and ψ and g be entire functions on $\mathbb{C}$. Then*

(i) $V_{(g,\psi)} : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^\infty$ *is bounded if and only if* $B^\infty_{(\psi,\alpha)}(|g|) \in L^\infty(dm)$. *Moreover, if $V_{(g,\psi)}$ is bounded, then its norm is estimated by*

$$\|V_{(g,\psi)}\| \simeq \sup_{z\in\mathbb{C}} B^\infty_{(\psi,\alpha)}(|g|)(z). \tag{2.1}$$

(ii) *A bounded map $V_{(g,\psi)} : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^\infty$ is compact if and only if*

$$\lim_{|\psi(z)|\to\infty} B^\infty_{(\psi,\alpha)}(|g|)(z) = 0. \tag{2.2}$$

(iii) $C_{(\psi,g)} : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^\infty$ *is bounded if and only if* $M^\infty_{(\psi,\alpha)}(|g|) \in L^\infty(dm)$. *Moreover, if $C_{(\psi,g)}$ is bounded, then its norm is estimated by*

$$\|C_{(\psi,g)}\| \simeq \sup_{z\in\mathbb{C}} M^\infty_{(\psi,\alpha)}(|g|)(z).$$

(iv) *A bounded map $C_{(\psi,g)} : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^\infty$ is compact if and only if*

$$\lim_{|\psi(z)|\to\infty} M^\infty_{(\psi,\alpha)}(|g|)(z) = 0.$$

It is interesting to note that both the conditions in (i) and (ii) are independent of the exponent p. It follows that if there exists a $p > 0$ for which $V_{(g,\psi)}$ is bounded (compact) from $\mathcal{F}_\alpha^p$ to $\mathcal{F}_\alpha^\infty$, then it is also bounded (compact) for every other p. This phenomena holds true for the operator $C_{(\psi,g)}$ as well.

As remarked earlier, when $\psi(z) = z$, the operators $V_{(g,\psi)}$ and $C_{(\psi,g)}$ reduce to the Volterra type integral operator V_g. Theorem 2.1 ensures that the symbol g can only grow as a power function of degree not exceeding respectively 2 and 1 to induce a bounded and compact V_g. We formulate this as follows.

Corollary 2.2. *Let $0 < p \leq \infty$. Then $V_g : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^\infty$ is*

(i) *bounded if and only if $g(z) = az^2 + bz + c$, for some $a, b, c \in \mathbb{C}$.*
(ii) *compact if and only if $g(z) = az + b$.*

The corollary extends similar representations obtained in [6, 15] for the symbol g.

Mapping $\mathcal{F}_\alpha^\infty$ into smaller spaces $\mathcal{F}_\alpha^p$ gives the next stronger condition.

Theorem 2.3. *Let $0 < p < \infty$, and ψ and g be entire functions on $\mathbb{C}$. Then the following are equivalent.*

(i) $V_{(g,\psi)} : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\alpha^p$ *is bounded;*
(ii) $V_{(g,\psi)} : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\alpha^p$ *is compact;*
(iii) $B_{(\psi,\alpha)}(|g|^p) \in L(dm)$. *Moreover, if $V_{(g,\psi)}$ is bounded, then*

$$\|V_{(g,\psi)}\|^p \simeq \int_{\mathbb{C}} B_{(\psi,\alpha)}(|g|^p)(w)dm(w). \tag{2.3}$$

Theorem 2.4. *Let $0 < p < \infty$, and ψ and g be entire functions on $\mathbb{C}$. Then the following are equivalent.*

(i) $C_{(\psi,g)} : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\alpha^p$ *is bounded;*
(ii) $C_{(\psi,g)} : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\alpha^p$ *is compact;*
(iii) $M_{(\psi,\alpha)}(|g|^p) \in L(dm)$. *Moreover if $C_{(\psi,g)}$ is bounded, then*

$$\|C_{(\psi,g)}\|^p \simeq \int_{\mathbb{C}} M_{(\psi,\alpha)}(|g|^p)(w)dm(w). \tag{2.4}$$

For the operators V_g and C_ψ, we in particular get the following explicit expressions for the inducing symbols.

Corollary 2.5. *Let $0 < p < \infty$ and ψ and g be entire functions on $\mathbb{C}$. Then*

(i) $V_g : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\alpha^p$ *is bounded (compact) if and only if $g(z) = az + b$ and $p > 2$ where a and b are in $\mathbb{C}$.*
(ii) $C_\psi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\alpha^p$ *is bounded (compact) if and only if $\psi(z) = az+b, |a| < 1$.*

Part (ii) extends the results in [4] where similar conditions are given for compact and bounded $C_\psi : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^q$ when $0 < p \leq q < \infty$ and in [15] when $0 < q < p < \infty$.

Proof. (i) We start by proving the necessity of the statements in (i). We first observe that by applying the explicit expression for the normalized reproducing kernels, we have

$$\begin{aligned} B_{(\psi,\alpha)}(|g|^p)(w) &= \int_{\mathbb{C}} \frac{|k_{(w,\alpha)}(\psi(z))g'(z)|^p}{\big((1+|z|)e^{\frac{\alpha}{2}|z|^2}\big)^p} dm(z) \\ &= \int_{\mathbb{C}} e^{\frac{p\alpha}{2}\left(2\Re\langle\psi(z),\ w\rangle - |z|^2 - |w|^2\right)} \frac{|g'(z)|^p}{(1+|z|)^p} dm(z) \\ &= \int_{\mathbb{C}} e^{\frac{p\alpha}{2}\left(|\psi(z)|^2 - |z|^2 - |w-\psi(z)|^2\right)} \frac{|g'(z)|^p}{(1+|z|)^p} dm(z) \end{aligned} \tag{2.5}$$

where the last equality follows by completing the square on the inner product. If $D(w,1) = \{z \in \mathbb{C} : |z-w| < 1\}$, then $1+|z| \simeq 1+|w|$ for each $z \in D(w,1)$.

It follows that

$$\begin{aligned}\int_{\mathbb{C}} B_{(\psi,\alpha)}(|g|^p)(w)dw &\geq \int_{\mathbb{C}}\int_{D(w,1)} \left|\frac{k_{(w,\alpha)}(z)g'(z)}{(1+|z|)}\right|^p e^{-\frac{p\alpha}{2}|z|^2} dm(z)dm(w)\\ &\geq \int_{\mathbb{C}}\int_{D(w,1)} \left|\frac{g'(w)}{1+|w|}\right|^p e^{-\frac{p\alpha}{2}|z-w|^2} dm(z)dm(w)\\ &\gtrsim \int_{\mathbb{C}} \left|\frac{g'(w)}{1+|w|}\right|^p dm(w).\end{aligned}$$

The desired restrictions on g and p follow once we apply Theorem 2.3. On the other hand, the sufficiency of the condition is immediate because by Theorem 2.3 again and Fubini's Theorem

$$\int_{\mathbb{C}} B_{(\psi,\alpha)}(|g|^p)(w)dm(w) \simeq \int_{\mathbb{C}}\int_{\mathbb{C}} \frac{e^{\frac{-p\alpha}{2}|w-z|^2}}{(1+|z|)^p} dm(z)dm(w) \simeq \int_{\mathbb{C}} \frac{dm(z)}{(1+|z|)^p} < \infty$$

for each $p > 2$.

(ii) Applying once again Theorem 2.3 and Fubini's Theorem, we obtain

$$\begin{aligned}\int_{\mathbb{C}}\int_{\mathbb{C}} \left|\frac{k_{(w,\alpha)}(\psi(z))}{e^{\frac{\alpha}{2}|z|^2}}\right|^p dm(z)dw(w) &= \int_{\mathbb{C}}\int_{\mathbb{C}} e^{\frac{p\alpha}{2}\left(|\psi(z)|^2-|z|^2-|\psi(z)-w|^2\right)} dm(w)dm(z)\\ &\simeq \int_{\mathbb{C}} e^{\frac{p\alpha}{2}(|\psi(z)|^2-|z|^2)} dm(z) < \infty\end{aligned}$$

if and only if ψ has a linear form as described above with $|a| < 1$. $\square$

2.2. The (∞, p) Fock–Carleson measures.

For $0 < p < \infty$, we call a nonnegative measure μ on $\mathbb{C}$ an (∞, p) Fock–Carleson measure if the inequality

$$\left(\int_{\mathbb{C}} |f(z)|^p e^{\frac{-p\alpha}{2}|z|^2} d\mu(z)\right)^{1/p} \lesssim \|f\|_{(\infty,\alpha)}$$

holds for all f in $\mathcal{F}_\alpha^\infty$. In other words, μ is an (∞, p) Fock–Carleson measure if and only if the embedding map $I_\mu : \mathcal{F}_\alpha^\infty \to L^p(d\mu)$ is bounded where $d\mu(z) = e^{-\frac{\alpha p}{2}|z|^2} dV(z)$. We call μ an (∞, p) vanishing Fock–Carleson measure when such an embedding map is compact. When both exponents are finite, the (p, q) Fock–Carleson measures were described in [9] in terms of the parameters t-Berezin transform of μ,

$$\widetilde{\mu_{(t,\alpha)}}(w) = \int_{\mathbb{C}} |k_{(w,\alpha)}(z)|^t e^{-\frac{\alpha t}{2}|z|^2} d\mu(z),$$

and $D(z, \delta)$, the Euclidean disc centered at z and radius $\delta > 0$. We will use these parameters to describe the (∞, p) Fock–Carleson measures as well.

We first remind a classical covering result that there exists a sequence of points z_k such that for any $r > 0$, the discs $D(z_k, r/2)$ cover $\mathbb{C}$ and the discs $D(z_k, r/4)$ are pairwise disjoint. From these conditions it follows that for any $\delta > 0$, there exists a positive integer $N_{\max} = N_{\max}(\delta, r)$ such that every point in $\mathbb{C}$ belongs to at most $N_{\max}$ of the discs. Then, we prove the

following result which is interesting by its own right apart from being needed to prove part of Theorem 2.3 and Theorem 2.4 in the next subsection.

Proposition 2.6. *Let $0 < p < \infty$, and $\mu > 0$ be a measure on $\mathbb{C}$. Then the following statements are equivalent.*

(i) *μ is an (∞, p) Fock–Carleson measure;*
(ii) *μ is an (∞, p) vanishing Fock–Carleson measure;*
(iii) *$\widetilde{\mu_{(t,\alpha)}} \in L(dm)$ for some (or any) $t > 0$;*
(iv) *μ is a finite measure on $\mathbb{C}$;*
(v) *$\mu(D(.,r)) \in L(dm)$ for some (or any) $r > 0$;*
(vi) *$\mu\big(D(z_k, r)\big) \in \ell^1$ for some (or any) $r > 0$. Moreover,*

$$\|I_\mu\|^p \simeq \|\widetilde{\mu_{(t,\alpha)}}\|_{L(dm)} \simeq \|\mu(D(.,\delta))\|_{L(dm)} \simeq \|\mu\big(D(z_k, r)\big)\|_{\ell^1}. \tag{2.6}$$

Proof. The equivalences of the statements in (iii), (v) and (vi) can be found in Lemma 2.3 of [9]. The proof of (i) implies (ii) follows from a simple modification of part of the arguments used in the proof of Theorem 3.3 in there again. Thus, we shall prove (i) implies (vi), (vi) implies (i), and (iii) implies (iv). For the first implication, we need to find a test function f_0 in F_α^∞ that would lead us to the desired conclusion. For this, we follow the classical Luecking's approach in [14]. First, we observe that for each c_j in ℓ^∞, by Theorem 8.2 in [10], we have that

$$f = \sum_{j=1}^{\infty} c_j k_{(z_j,\alpha)} \in F_\alpha^\infty \text{ and } \|f\|_{(\infty,\alpha)} \lesssim \|(c_j)\|_{\ell^\infty}.$$

Taking $c_k = 1$ for all k does the job for our case here. Thus we set our test function as

$$f_0 = \sum_{j=1}^{\infty} k_{(z_j,\alpha)}.$$

Since μ is an (∞, p) Fock–Carleson measure,

$$\int_{\mathbb{C}} |f_0(z)|^p e^{-\frac{\alpha p}{2}|z|^2} d\mu(z) \le \|I_\mu\|^p \|f_0\|^p_{(\infty,\alpha)} \lesssim \|I_\mu\|^p.$$

If $r_j's(t)$ are the Rademacher sequence of functions on $[0, 1]$ chosen as in [14], then Khinchine's inequality yields

$$\left(\sum_{j=1}^{\infty} |k_{(z_j,\alpha)}(z)|^2\right)^{p/2} \lesssim \int_0^1 \left|\sum_{j=1}^{\infty} r_j(t) k_{(z_j,\alpha)}(z)\right|^p dt. \tag{2.7}$$

Note that if the $r_j(t)$ are chosen as refereed above, then $r_j(t)$ belongs to ℓ^∞ and $\|(r_j(t))\|_{\ell^\infty} \simeq 1$. Moreover, it holds that

$$\sum_{j=1}^{\infty} r_j(t) k_{(z_j,\alpha)} \in F_\alpha^\infty, \text{ with } \Big\|\sum_{j=1}^{\infty} r_j(t) k_{(z_j,\alpha)}\Big\|_{(\infty,\alpha)} \simeq 1.$$

Making use of first (2.7) and subsequently Fubini's theorem, we obtain

$$\int_{\mathbb{C}}\left(\sum_{k=1}^{\infty}|k_{(z_j,\alpha)}|^2\right)^{\frac{p}{2}}d\mu(z) \lesssim \int_{\mathbb{C}}\left(\int_0^1\left|\sum_{j=1}^{\infty}r_j(t)k_{(z_j,\alpha)}(z)\right|^p dt\right)d\mu(z)$$
$$= \int_0^1\left(\int_{\mathbb{C}}\left|\sum_{j=1}^{\infty}r_j(t)k_{(z_j,\alpha)}(z)\right|^p d\mu(z)\right)dt \lesssim \|I_\mu\|^p. \quad (2.8)$$

On the other hand,

$$\sum_{j=1}^{\infty}\mu(D(z_j,r)) = \int_{\mathbb{C}}\sum_{j=1}^{\infty}\chi_{D(z_j,r)}(z)d\mu(z) \leq \int_{\mathbb{C}}\left(\sum_{j=1}^{\infty}\chi_{D(z_j,r)}(z)\right)^{p/2}d\mu(z)$$

which follows by Hölder's inequality for $p<2$ and general fact for $p\geq 2$ as all the terms are positive. The last integral above is bounded by

$$\int_{\mathbb{C}}\left(\sum_{j=1}^{\infty}e^{\alpha r^2}|k_{(z_j,\alpha)}|^2\right)^{p/2}d\mu(z) \lesssim \int_{\mathbb{C}}\left(\sum_{j=1}^{\infty}|k_{(z_j,\alpha)}|^2\right)^{p/2}d\mu(z).$$

This combined with (2.8) gives

$$\sum_{j=1}^{\infty}\mu(D(z_j,r)) \lesssim \|I_\mu\|^p. \quad (2.9)$$

To prove that (i) follows from (vi), by Lemma 1 of [12], we observe that

$$\sup_{z\in D(z_k,r)}|f(z)e^{-\frac{\alpha}{2}|z|^2}|^p \lesssim \int_{D(z_k,2r)}|f(z)e^{-\frac{\alpha}{2}|z|^2}|^p dm(z)$$

for each k. Thus, we get

$$\int_{\mathbb{C}}|f(z)e^{-\frac{\alpha}{2}|z|^2}|^p d\mu(z) \leq \sum_{k=1}^{\infty}\int_{D(z_k,r)}|f(z)e^{-\frac{\alpha}{2}|z|^2}|^p d\mu(z)$$
$$\lesssim \|\mu(D(z_k,r))\|_{\ell^1}\sum_{k=1}^{\infty}\int_{D(z_k,2r)}|f(z)e^{-\frac{\alpha}{2}|z|^2}|^p dm(z)$$
$$\lesssim \|\mu(D(z_k,r))\|_{\ell^1}\|f\|^p_{(\infty,\alpha)}$$

from which the estimate

$$\|I_u\|^p \lesssim \|\mu(D(z_k,r))\|_{\ell^1} \quad (2.10)$$

holds. Now we combine (2.10), (2.9) and the estimates in Theorem 3.3 of [9] to get all the remaining norm estimates in (2.6).

We remain to show (iii) implies (iv). But this global geometric condition follows when we in particular set $t=1$. Because by Fubini's theorem, we

may have

$$\int_{\mathbb{C}} \widetilde{\mu}_{(1,\alpha)}(z)dm(z) = \int_{\mathbb{C}}\int_{\mathbb{C}} e^{\frac{\alpha}{2}|\langle w,z\rangle|^2-\frac{\alpha}{2}|z|^2-\frac{\alpha}{2}|w|^2} d\mu(w)dm(z)$$
$$= \int_{\mathbb{C}}\left(\int_{\mathbb{C}} e^{-\frac{\alpha}{2}|z-w|^2}dm(z)\right)d\mu(w) \simeq \mu(\mathbb{C}).$$

2.3. Proof of the main results.

Using a result of H. Cho and K. Zhu [5], recently Constantin [6] proved that

$$\int_{\mathbb{C}} |f(z)|^p e^{-p\alpha|z|^2/2}dm(z) \simeq |f(0)|^p + \int_{\mathbb{C}} \frac{|f'(z)|^p}{(1+|z|)^p} e^{-p\alpha|z|^2/2}dm(z) \quad (2.11)$$

holds for each entire function f and $0 < p < \infty$. From a simple variant of the arguments used in the proof of (2.11), we also conclude that

$$\sup_{z\in\mathbb{C}} |f(z)|e^{\frac{-\alpha}{2}|z|^2} \simeq |f(0)| + \sup_{z\in\mathbb{C}} \frac{|f'(z)|}{1+|z|}e^{\frac{-\alpha}{2}|z|^2} \quad (2.12)$$

for each entire function f. Observe that (2.11) and (2.12) describe the Fock spaces in terms of derivatives. Such a description plays an important role in our further analysis. In fact, it makes our analysis easier by eliminating the integral that arises from the Volterra type integral operator V_g. By Lemma 1 of [19], we also have the pointwise estimate

$$|f(z)|e^{-\frac{\alpha}{2}|z|^2} \leq \|f\|_{(p,\alpha)} \quad (2.13)$$

for each point z in $\mathbb{C}$, $0 < p \leq \infty$ and f in F_α^p. From this along with (2.12) it follows that

$$\|V_{(g,\psi)}f\|_{(\infty,\alpha)} \simeq \sup_{z\in\mathbb{C}} \frac{|g'(z)|}{1+|z|}|f(\psi(z))|e^{\frac{-\alpha}{2}|z|^2} \leq \|f\|_{(p,\alpha)} \sup_{z\in\mathbb{C}} B^\infty_{(\psi,\alpha)}(|g|)(z)$$

from which the sufficiency of the statement in part (i) of Theorem 2.1 and the estimate

$$\|V_{(g,\psi)}\| \lesssim \sup_{z\in\mathbb{C}} B^\infty_{(\psi,\alpha)}(|g|)(z) \quad (2.14)$$

follow. To prove its necessity, we apply $V_{(g,\psi)}$ to the normalized kernel function $k_{(w,\alpha)}$ and estimate the resulting function (∞,α) norm using (2.12) as

$$\begin{aligned} \|V_{(g,\psi)}k_{(w,\alpha)}\|_{(\infty,\alpha)} &\simeq \sup_{z\in\mathbb{C}} \frac{|g'(z)|}{1+|z|}|k_{(w,\alpha)}(\psi(z))|e^{\frac{-\alpha}{2}|z|^2} \\ &\geq \frac{|g'(z)|}{1+|z|}|k_{(w,\alpha)}(\psi(z))|e^{\frac{-\alpha}{2}|z|^2} \end{aligned}$$

for all $z, w \in \mathbb{C}$. In particular, when we set $w = \psi(z)$, we get

$$\|V_{(g,\psi)}k_{(w,\alpha)}\|_{(\infty,\alpha)} \gtrsim B^\infty_{(\psi,\alpha)}(|g|)(z)$$

from which the reverse inequality in (2.14) holds.

To prove the second part of the theorem, we extend a technique used in [19, 21]. We first assume that $V_{(g,\psi)}$ is compact. The sequence $k_{(w,\alpha)}$ converges to zero as $|w| \to \infty$, and the convergence is uniform on compact subset of

$\mathbb{C}$. We further assume that there exists sequence of points $z_j \in \mathbb{C}$ such that $|\psi(z_j)| \to \infty$ as $j \to \infty$. If such a sequence does not exist, then necessity holds trivially. It follows from compactness of $V_{(g,\psi)}$ that

$$\limsup_{j\to\infty} B^{\infty}_{(\psi,\alpha)}(z) \le \limsup_{j\to\infty} \|V_{(g,\psi)}k_{(w,\alpha)}\|_{(\infty,\alpha)} = 0 \tag{2.15}$$

from which (2.2) follows.

We now assume that condition (2.2) holds, and proceed to show that $V_{(g,\psi)}$ is a compact map. The function $f(z) = 1$ belongs to $\mathcal{F}^p_\alpha$ for all $p > 0$. It follows that by boundedness,

$$\sup_{z\in\mathbb{C}} \frac{|g'(z)|}{1+|z|} e^{-\frac{\alpha}{2}|z|^2} < \infty. \tag{2.16}$$

Let f_j be a sequence of functions in $\mathcal{F}^p_\alpha$ such that $\sup_m \|f_j\|_{(p,\alpha)} < \infty$ and f_j converges uniformly to zero on compact subsets of $\mathbb{C}$ as $j \to \infty$. For each $\epsilon > 0$, condition (2.2) implies that there exists a positive N_1 such that $B^{\infty}_{(\psi,\alpha)}(z) < \epsilon$ for all $|\psi(z)| > N_1$. From this together with (2.13), we obtain

$$\frac{|g'(z)|}{1+|z|}|f_j(\psi(z))|e^{-\frac{\alpha}{2}|z|^2} \le \|f_j\|_{(p,\alpha)} \frac{|g'(z)|}{1+|z|} e^{\frac{\alpha}{2}|\psi(z)|^2-\frac{\alpha}{2}|z|^2} \lesssim \epsilon \tag{2.17}$$

for all $|\psi(z)| > N_1$ and all j. On the other hand if $|\psi(z)| \le N_1$, then applying (2.16) it is easily seen that

$$\frac{|g'(z)|}{1+|z|}|f_j(\psi(z))|e^{-\frac{\alpha}{2}|z|^2} \lesssim \sup_{z:|\psi(z)|\le N_1} |f_j(\psi(z))| \to 0 \tag{2.18}$$

as $j \to \infty$. Then we apply (2.17),(2.18) and (2.12) to arrive at the desired conclusion.
(iii) We first note that $(C_{(\psi,g)}f(z))' = g'(\psi(z))\psi'(z)f(\psi(z))$. This together with (2.12) implies

$$\begin{aligned}\|C_{(\psi,g)}f\|_{(\infty,\alpha)} &\simeq \sup_{z\in\mathbb{C}} \frac{|g'(\psi(z))\psi'(z)|}{1+|z|}|f(\psi(z))|e^{\frac{-\alpha}{2}|z|^2}\\ &\lesssim \|f\|_{(p,\alpha)} \sup_{z\in\mathbb{C}} \frac{|g'(\psi(z))\psi'(z)|}{1+|z|} e^{\frac{\alpha}{2}|\psi(z)|^2-\frac{\alpha}{2}|z|^2}\\ &= \|f\|_{(p,\alpha)} \sup_{z\in\mathbb{C}} M^{\infty}_{(\psi,\alpha)}(|g|)(z)\end{aligned}$$

where the second inequality follows by (2.13). From this the sufficiency part of (iii) and the estimate

$$\|C_{(\psi,g)}\| \lesssim \sup_{z\in\mathbb{C}} M^{\infty}_{(\psi,\alpha)}(|g|)(z) \tag{2.19}$$

follow. To prove the necessity, we apply $C_{(\psi,g)}$ to the normalized kernel function again and observe that

$$\|C_{(\psi,g)}k_{(w,\alpha)}\|_{(\infty,\alpha)} \gtrsim \frac{|g'(\psi(z))\psi'(z)|}{1+|z|}|k_{(w,\alpha)}(\psi(z))|e^{\frac{-\alpha}{2}|z|^2}$$

for all $z, w \in \mathbb{C}$. In particular when we set $w = \psi(z)$, we again get

$$\|C_{(\psi,g)} k_{(w,\alpha)}\|_{(\infty,\alpha)} \gtrsim \sup_{z\in\mathbb{C}} M^{\infty}_{(\psi,\alpha)}(|g|)(z)$$

from which the reverse inequality in (2.19) also follows.
The proof of part (iv) of the theorem is very similar to the proof of part (ii). We thus omit it.
Proof of Theorem 2.3. For each $p > 0$, we set $\mu_{(p,\alpha)}$ to be the positive pull back measure on $\mathbb{C}$ defined by

$$\mu_{(p,\alpha)}(E) = \int_{\psi^{-1}(E)} \frac{|g'(z)|^p}{(1+|z|)^p} e^{-\frac{p\alpha}{2}|z|^2} dm(z)$$

for every Borel subset E of $\mathbb{C}$. By (2.11) and substitution, we observe that $V_{(g,\psi)} : \mathcal{F}^{\infty}_{\alpha} \to \mathcal{F}^{p}_{\alpha}$ is bounded if and only if

$$\begin{aligned}\int_{\mathbb{C}} |V_{(g,\psi)} f(z)|^p e^{\frac{-p\alpha}{2}|z|^2} dm(z) &\simeq \int_{\mathbb{C}} |f(z)|^p d\mu_{(p,\alpha)}(z)\\ &= \int_{\mathbb{C}} |f(z)|^p e^{\frac{-p\alpha}{2}|z|^2} d\lambda_{(p,\alpha)}(z) \lesssim \|f\|^p_{(\infty,\alpha)}\end{aligned}$$

where $d\lambda_{(p,\alpha)}(z) = e^{\frac{p\alpha}{2}|z|^2} d\mu_{(p,\alpha)}(z)$. The inequality above holds if and only if $\lambda_{(p,\alpha)}$ is an (∞, p) Fock–Carleson measure. By a similar analysis, it is easy to see that $V_{(g,\psi)} : F^{\infty}_{\alpha} \to F^{p}_{\alpha}$ is compact if and only if the measure $\lambda_{(p,\alpha)}$ is an (∞, p) vanishing Fock–Carleson measure. By Proposition 2.6, it follows that the statements in (i) and (ii) are equivalent, and any one of them holds if and only

$$\widetilde{\lambda_{(P,\alpha)}}(w) = \int_{\mathbb{C}} |k_{(w,\alpha)}(z)|^p e^{\frac{-p\alpha}{2}|z|^2} d\lambda_{(p,\alpha)}(z) \in L(\mathbb{C}, dm)$$

from which substituting back $d\lambda_{(p,\alpha)}$ and $d\mu_{(p,\alpha)}$ in terms of dm, we obtain

$$\widetilde{\lambda_{(p,\alpha)}}(w) = \int_{\mathbb{C}} |k_{(w,\alpha)}(z)|^p e^{\frac{-p\alpha}{2}|z|^2} d\lambda_{(p,\alpha)}(z) \simeq B_{(\psi,\alpha)}(|g|^p)(w).$$

We remain to prove the norm estimate in (iii). But this rather can be deduce easily. Since $\lambda_{(p,\alpha)}$ is an (∞, p) Fock–Carleson measure, the series of norm estimates in Theorem 2.6 yield

$$\|V_{(g,\psi)}\|^p = \|I_{\lambda_{(p,\alpha)}}\|^p \simeq \|\widetilde{\lambda_{(p,\alpha)}}\|_{L(\mathbb{C},dm)} \simeq \|B_{(\psi,\alpha)}(|g|^p)\|_{L(\mathbb{C},dm)}$$

and completes the proof of the theorem.
Proof of Theorem 2.4. Applying (2.2), we compute

$$\begin{aligned}\|C_{(\psi,g)}f\|_{(p,\alpha)}^p &\simeq \int_{\mathbb{C}} |(C_{(\psi,g)}f)'(z)|^p(1+|z|)^{-p}e^{\frac{-p\alpha}{2}|z|^2}dm(z)\\ &= \int_{\mathbb{C}} \frac{|f(\psi(z))g'(\psi(z))|^p}{(1+|z|)^p|\psi'(z)|^{-p}}e^{\frac{-p\alpha}{2}|z|^2}dm(z)\\ &= \int_{\mathbb{C}} |f(\psi(z))|^p e^{\frac{-p\alpha}{2}|\psi(z)|^2}dv(z)\\ &= \int_{\mathbb{C}} |f(z)|^p e^{\frac{-p\alpha}{2}|z|^2}d\theta(z)\end{aligned}$$

where $d\theta(z) = dv(\psi^{-1}(z))$ and

$$dv(w) = \frac{|g'(\psi(w))|^p e^{\frac{p\alpha}{2}\left(|\psi(w)|^2-|w|^2\right)}}{(1+|w|)^p|\psi'(w)|^{-p}}dm(w).$$

From this it follows that the map $C_{(\psi,g)} : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\alpha^p$ is bounded (compact) if and only if θ is an (∞, p) (vanishing) Fock–Carleson measure. The remaining part of the proof is very similar to the proof of the last part of Theorem 2.3 above and we omit it.

We note in passing that using the above simple techniques of reformulating the bounded and compact properties of $C_{(\psi,g)}$ in terms of Fock–Carleson measures, Theorem 3.1, Theorem 3.2 and Theorem 3.3 of [9], we deduce the following.

Proposition 2.7. *Let $0 < p \leq q < \infty$, and ψ and g be entire functions on $\mathbb{C}$. Then*

(i) $C_{(\psi,g)} : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^q$ *is bounded if and only if* $M_{(\psi,\alpha)}(|g|^q) \in L^\infty(\mathbb{C}, dm)$, *with norm estimated by*

$$\|C_{(\psi,g)}\|^q \simeq \sup_{w\in\mathbb{C}} M_{(\psi,\alpha)}(|g|^q)(w).$$

(ii) $C_{(\psi,g)} : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^q$ *compact if and only if*

$$\lim_{|z|\to\infty} M_{(\psi,\alpha)}(|g|^q)(z) = 0.$$

(iii) $C_{(\psi,g)} : \mathcal{F}_\alpha^q \to \mathcal{F}_\alpha^p$, $p \neq q$, *is bounded (compact) if and only if* $M_{(\psi,\alpha)}(|g|^p) \in L^{q/(q-p)}(\mathbb{C}, dm)$, *with norm estimated by ,*

$$\|C_{(\psi,g)}\|^{\frac{q}{q-p}} \simeq \int_{\mathbb{C}} M_{(\psi,\alpha)}^{\frac{q}{q-p}}(|g|^p)(w)dm(w).$$

3. Weighted Composition operator

Each pair of entire functions (u, ψ) induces a weighted composition operator defined by $(uC_\psi)f = u.f(\psi)$. The various mapping properties of uC_ψ acting on a number of spaces of holomorphic function have been studied by several authors. Its bounded and compact properties were studied in [19, 21] when

it acts from $\mathcal{F}_\alpha^p$ to $\mathcal{F}_\alpha^\infty$ and $0 < p \leq \infty$. But the problem to identify these properties has been open when uC_ψ acts from $\mathcal{F}_\alpha^\infty$ to smaller spaces $\mathcal{F}_\alpha^p$, $0 < p < \infty$ as far as we know. In this section, we will answer this problem by relating it with some mapping properties of $V_{(g,\psi)}$.

Interestingly, the description of Fock spaces in terms of derivative, (2.11) and (2.12), makes it possible to link some operator theoretic results on product of Volterra type integral and composition operators $V_{(g,\psi)}$ with weighted composition operators uC_ψ when all acting between Fock spaces. For example; the bounded and compactness results on $uC_\psi : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^\infty$ in [19, 21] could be directly read by simply replacing the weight $|g'(z)|/(1 + |z|)$ by $|u(z)|$ in the conditions of Theorem 2.1. In fact, the inequalities in (2.1), which arise because of (2.12), will be replaced by equality for the case of weighted composition operators. By making the same replacement in the conditions of Theorem 2.3, we deduce the following result and answer the open problem mentioned in the previous paragraph.

Corollary 3.1. *Let $0 < p < \infty$, and ψ and g be entire functions on $\mathbb{C}$. Then the following statements are equivalent.*

(i) $uC_\psi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\alpha^p$ *is bounded;*
(ii) $uC_\psi : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\alpha^p$ *is compact;*
(iii)

$$\int_{\mathbb{C}} \int_{\mathbb{C}} |k_{(w,\alpha)}(\psi(z))u(z)|^p e^{-\frac{p\alpha}{2}|z|^2} dm(z)dm(w) < \infty.$$

Moreover, if uC_ψ is bounded, then we have the asymptotic estimate

$$\|uC_\psi\|^p \simeq \int_{\mathbb{C}} \int_{\mathbb{C}} |k_{(w,\alpha)}(\psi(z))u(z)|^p e^{-\frac{p\alpha}{2}|z|^2} dm(z)dm(w) < \infty.$$

We note that because of the derivative on g and the factor $1/(1 + |z|)$, more symbols g are admissible for bounded and compact $V_{(g,\psi)}$ than the corresponding weight u for bounded and compact uC_ψ.

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